JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 23, No. 6, November–December 2000

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Analytic Steady-State Accuracy of a Spacecraft Attitude Estimator

F. Landis Markley*
NASA Goddard Space Flight Center, Greenbelt, Maryland 20771
and
R. G. Reynolds†
SpectrumAstro, Inc., Herndon, Virginia 20170

Introduction

THERE has long been an interest in Kalman filtering[1,2] to combine optimally star tracker and gyro data for spacecraft attitude estimation.[3,4] In particular, Farrenkopf's analytic solution for the steady-state accuracy of a single-axis Kalman filter combining data from a gyro and an angle sensor has proven very useful for preliminary analysis of spacecraft attitude determination systems.[5] Farrenkopf implicitly assumed a rate gyro model, however, and many spacecraft employ rate-integrating gyros (RIGs) that exhibit angle output white noise (also known as readout noise or electronic noise).[6] The aim of this Note is to modify Farrenkopf's results[5] to include the effects of gyro angle output noise.

We assume that we take gyro measurements at a time interval τ and star sensor measurements at a multiple $T = n\tau$ of this, with each star tracker measurement immediately following a gyro measurement. Consider a star tracker measurement update at some time arbitrarily labeled zero, after attainment of steady state. Let $\hat{\boldsymbol{x}}_0(\mp)$ and $P_0(\mp)$ denote the state estimate and its covariance immediately before and after this star tracker update, respectively. Similarly, let $\hat{\boldsymbol{x}}_k(+)$ and $P_k(+)$ for $k = 1, 2, \ldots, n$ denote the state estimate and covariance at time $k\tau$ incorporating the information from the gyro measurement at that time. The steady-state condition is that the covariance after processing all these measurements, $P_n(+)$, be identical with the covariance $P_0(-)$ before the star tracker update but after the gyro measurement update immediately preceding it.

Spacecraft and Gyro Dynamics

The single-axis spacecraft kinematics are given by

$$\dot{\theta} = \omega \tag{1}$$

where θ is the rotation angle and ω is the true angular velocity. Because of its own internal dynamics, the RIG does not measure θ exactly, but instead accumulates an angle ϕ by

$$\dot{\phi} = \omega + b + n_v \tag{2}$$

Received 13 April 2000; revision received 15 June 2000; accepted for publication 15 June 2000.

*Aerospace Engineer, Guidance, Navigation, and Control Center. Fellow AIAA.
†Senior ACS Engineer. Member AIAA.

where b is the gyro drift rate and n_v is a zero mean Gaussian white noise process. Thus, we are led to consider the three-component state vector

$$\boldsymbol{x} = [\theta, b, \phi]^T \tag{3}$$

which obeys the dynamic equation

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + \omega \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ n_u \\ n_v \end{bmatrix} \tag{4}$$

The gyro noise processes n_u and n_v are assumed to be independent with zero mean and with autocorrelation functions

$$E\{n_u(t)n_u(t')\} = \sigma_u^2 \delta(t - t') \tag{5a}$$

$$E\{n_v(t)n_v(t')\} = \sigma_v^2 \delta(t - t') \tag{5b}$$

where $E\{\cdot\}$ denotes the expectation value and $\delta(\cdot)$ is the Dirac delta function. The finite time propagation of the state from time $(k-1)\tau$ to $k\tau$ is given by

$$\boldsymbol{x}_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \tau & 1 \end{bmatrix} \boldsymbol{x}_{k-1} + u_k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_k \\ \beta_k \end{bmatrix} \tag{6}$$

where

$$u_k \equiv \int_{(k-1)\tau}^{k\tau} \omega(t)\,\mathrm{d}t \tag{7a}$$

$$\alpha_k \equiv \int_{(k-1)\tau}^{k\tau} n_u(t)\,\mathrm{d}t \tag{7b}$$

$$\beta_k \equiv \int_{(k-1)\tau}^{k\tau} n_v(t)\,\mathrm{d}t + \int_{(k-1)\tau}^{k\tau} (k\tau - t)\,n_u(t)\,\mathrm{d}t \tag{7c}$$

We want to use the RIG measurements rather than a dynamic model for angular velocity information. The measurement at time $k\tau$ is given by

$$\tilde{\phi}_k = [0, 0, 1]\boldsymbol{x}_k + v_k \tag{8}$$

where v_k is the zero mean gyro angle output white noise, which is assumed to obey

$$E\{v_k v_l\} = \sigma_e^2 \delta_{kl} \tag{9}$$

with δ_{kl} the Kronecker delta. Substituting $\boldsymbol{x}_k$ from Eq. (6) into Eq. (8) and rearranging terms give

$$u_k = -[0, \tau, 1]\boldsymbol{x}_{k-1} + \tilde{\phi}_k - \beta_k - v_k \tag{10}$$

Substituting this expression for u_k on the right side of Eq. (6) effectively replaces the unknown spacecraft rotational dynamics by the RIG measurement in the system dynamics, giving

$$\boldsymbol{x}_k = \Phi(\tau)\boldsymbol{x}_{k-1} + \tilde{\phi}_k[1, 0, 1]^T + \boldsymbol{w}_k \tag{11}$$

where

$$\Phi(\tau) \equiv \begin{bmatrix} 1 & -\tau & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

$$w_k \equiv [-\beta_k - \nu_k, \alpha_k, -\nu_k]^T \tag{13}$$

Equation (11), which includes both the time propagation and the gyro measurement update, no longer depends explicitly on the unknown spacecraft rotational dynamics. The corresponding covariance propagation equation is

$$P_k(+) = \Phi(\tau)P_{k-1}(+)\Phi^T(\tau) + Q(\tau) \tag{14}$$

where

$$Q(\tau) \equiv E\{w_k w_k^T\} = \begin{bmatrix} \tau\sigma_v^2 + \frac{1}{3}\tau^3\sigma_u^2 + \sigma_e^2 & -\frac{1}{2}\tau^2\sigma_u^2 & \sigma_e^2 \\ -\frac{1}{2}\tau^2\sigma_u^2 & \tau\sigma_u^2 & 0 \\ \sigma_e^2 & 0 & \sigma_e^2 \end{bmatrix} \tag{15}$$

It is easy to show by induction on Eq. (14) that

$$P_k(+) = \Phi(k\tau)P_0(+)\Phi^T(k\tau) + Q(k\tau) \tag{16}$$

for $k = 1, 2, \ldots, n$, where $Q(k\tau)$ denotes the matrix in Eq. (15) with τ replaced by $k\tau$. Note that the covariance depends only on the total elapsed time and not on the gyro update interval τ.

Star Tracker Measurement and Steady-State Covariance

The covariance update due to the star tracker measurement of the angle θ at time zero is

$$P_0(+) = P_0(-) - P_0(-)H_{st}^T\left[H_{st}P_0(-)H_{st}^T + \sigma_n^2\right]^{-1}H_{st}P_0(-) \tag{17}$$

where

$$H_{st} = [1, 0, 0] \tag{18}$$

and σ_n^2 is the variance of the star tracker white noise. We find the steady-state angle and drift rate covariances by inserting Eq. (17) into Eq. (16) for $k = n$ and by using the steady-state condition that $P_n(+) = P_0(-)$. It is convenient to define the dimensionless parameters S_e, S_u, S_v, and ζ by

$$S_e \equiv \sigma_e/\sigma_n \tag{19a}$$

$$S_u \equiv T^{\frac{3}{2}}\sigma_u\big/\sigma_n \tag{19b}$$

$$S_v \equiv T^{\frac{1}{2}}\sigma_v\big/\sigma_n \tag{19c}$$

$$P_{\theta b}(-) = -T^{\frac{1}{2}}\sigma_u\sigma_n\zeta \tag{20}$$

In Eq. (20) and in the following, we label components of $P_0(\pm)$ by variable names rather than numerically and suppress the time index 0 for convenience. With these definitions, we find that

$$P_{\theta\theta}(-) = (\zeta^2 - 1)\sigma_n^2 \tag{21a}$$

$$P_{\theta\theta}(+) = (1 - \zeta^{-2})\sigma_n^2 \tag{21b}$$

$$P_{bb}(\mp) = \left[\zeta - \left(1 + S_e^2\right)\zeta^{-1}\right]T^{-\frac{1}{2}}\sigma_u\sigma_n \pm \tfrac{1}{2}T\sigma_u^2 \tag{22}$$

where ζ is a solution of the quartic equation

$$\left[\zeta^2 - 2\left(\gamma + \tfrac{1}{4}S_u\right)\zeta + 1 + S_e^2\right]\left[\zeta^2 + 2\left(\gamma - \tfrac{1}{4}S_u\right)\zeta + 1 + S_e^2\right] = 0 \tag{23}$$

with

$$\gamma \equiv \left(1 + S_e^2 + \tfrac{1}{4}S_v^2 + \tfrac{1}{48}S_u^2\right)^{\frac{1}{2}} \geq 1 \tag{24}$$

The only root of Eq. (23) giving a positive-definite covariance is the largest root,

$$\zeta = \gamma + \tfrac{1}{4}S_u + \tfrac{1}{2}\left(2\gamma S_u + S_v^2 + \tfrac{1}{3}S_u^2\right)^{\frac{1}{2}} \tag{25}$$

This can be used to rewrite Eq. (22) in the more convenient form

$$P_{bb}(\mp) = \left(2\gamma T^{\frac{1}{2}}\sigma_u\sigma_n + \sigma_v^2 + \tfrac{1}{3}T^2\sigma_u^2\right)^{\frac{1}{2}}\sigma_u \pm \tfrac{1}{2}T\sigma_u^2 \tag{26}$$

The principal results of this Note are given by Eqs. (21) and (24–26), which differ from Farrenkopf's[5] only by details of notation and the S_e^2 term in Eq. (24). For $\sigma_e = 0$, these equations are completely equivalent to Farrenkopf's.

Numerical Example

Consider a RIG with $\sigma_v = 0.025$ deg/$\sqrt{\text{h}}$ = 7.27 μrad/$\sqrt{\text{s}}$, $\sigma_u = 3.7 \times 10^{-3}$ deg/h$^{3/2}$ = 3×10^{-4} μrad/s$^{3/2}$, and $\sigma_e = 15$ μrad. These numbers are characteristic of a ring-laser gyro with very low drift but with significant angle white noise. Assume that the star tracker measurement noise is $\sigma_n = 15$ μrad. Figure 1 shows the steady-state pre-update and post-update angle standard deviations, the square roots of $P_{\theta\theta}(\mp)$, for star tracker update times between 0.01 and 100 s. The solid curves are for the specified value of σ_e, and the dashed curves are for $\sigma_e = 0$. For each pair of curves, the upper curve is the pre-update value, and the lower curve is the post-update value. The limits of the pre-update and post-update angle standard deviations as the update time goes to zero are σ_e and $\sigma_e\sigma_n/\sqrt{(\sigma_e^2 + \sigma_n^2)}$, respectively. It is clear from Fig. 1 and from the analytic form for the limit that increasing the frequency of star tracker updates has less effect on the angle estimation accuracy when gyro output white noise is present than it has in the absence of this error source.

The steady-state drift bias standard deviations are not plotted because both the pre-update and post-update values always lie between 0.0467 and 0.0468 μrad/s. To this accuracy, the lower value is equal to $\sqrt{(\sigma_u\sigma_v)}$, the limit of Eq. (26) for small σ_u.

State Update Equations

The transformation of the state estimates over one time step are found by taking the expectation values of Eq. (11), noting that w_k has zero mean. The results in component form are

$$\hat{\theta}_k(+) = \hat{\theta}_{k-1}(+) + \tilde{\phi}_k - \hat{\phi}_{k-1}(+) - \tau\hat{b}_{k-1}(+) \tag{27a}$$

$$\hat{b}_k(+) = \hat{b}_{k-1}(+) \tag{27b}$$

$$\hat{\phi}_k(+) = \tilde{\phi}_k \tag{27c}$$

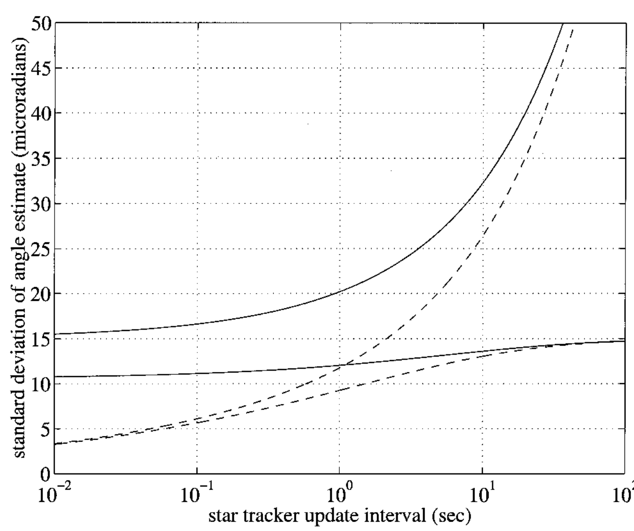

Fig. 1 Steady-state preupdate and postupdate angle standard deviations, solid curves for σ_e = 15 θrad, dashed curves for σ_e = 0.

Equations (27b) and (27c) show that the estimate of the bias is not refined by the gyro measurement, as expected, and that the estimate of the gyro-sensed angle is equal to the gyro measurement itself. Equation (27a) has a straightforward interpretation for $k \geq 2$. Because $\hat{\phi}_{k-1}(+) = \tilde{\phi}_{k-1}$ in this case, the estimate of the spacecraft rotation angle is the preceding estimate plus the rotation sensed by the gyro, corrected for the estimated drift bias.

The steady-state gains for the star tracker update are given by

$$K_{\text{st}} = P_0(+)H_{\text{st}}^T\sigma_n^{-2} = (\zeta\sigma_n)^{-2}[P_{\theta\theta}(-), P_{\theta b}(-), P_{\theta\phi}(-)]^T \tag{28}$$

The components of the state vector are updated by

$$\hat{\theta}_0(+) = \hat{\theta}_0(-) + (1 - \zeta^{-2})[\tilde{\theta}_0 - \hat{\theta}_0(-)] \tag{29a}$$

$$\hat{b}_0(+) = \hat{b}_0(-) - (\zeta T)^{-1}S_u[\tilde{\theta}_0 - \hat{\theta}_0(-)] \tag{29b}$$

$$\hat{\phi}_0(+) = \hat{\phi}_0(-) + (S_e/\zeta)^2[\tilde{\theta}_0 - \hat{\theta}_0(-)] \tag{29c}$$

where $\tilde{\theta}_0$ is the observed value. It is convenient to define a modified angle estimate $\bar{\theta}$ by

$$\bar{\theta}_k(+) \equiv \hat{\theta}_k(+) + \tilde{\phi}_k - \hat{\phi}_k(+) \tag{30}$$

for $k = 0, 1, 2, \ldots, n$, where $\tilde{\phi}_0$ is understood to denote the gyro measurement immediately preceding the star tracker measurement at time zero. Equation (27c) shows that $\bar{\theta}$ is the optimal angle estimate following a gyro update, but Eq. (29c) shows that it differs from the optimal estimate after a star tracker update. Because $\hat{\theta}_0(-)$ is the angle estimate following the gyro update immediately preceding the star tracker measurement, it is consistent with our conventions to denote this quantity by $\bar{\theta}_0(-)$. With this notation, the gyro propagation of the attitude can be written as

$$\bar{\theta}_k(+) = \bar{\theta}_{k-1}(+) + \tilde{\phi}_k - \tilde{\phi}_{k-1} - \tau\hat{b}_{k-1}(+) \tag{31}$$

and the star tracker update of the angle estimate can be performed as a two-step process

$$\bar{\theta}_0(+) = \bar{\theta}_0(-) + \zeta^{-1}\left(2\gamma S_u + S_v^2 + \tfrac{1}{3}S_u^2\right)^{\frac{1}{2}}[\tilde{\theta}_0 - \bar{\theta}_0(-)] \tag{32a}$$

followed by

$$\hat{\theta}_0(+) = \left(\sigma_n^2 + \sigma_e^2\right)^{-1}\left[\sigma_n^2\bar{\theta}_0(+) + \sigma_e^2\tilde{\theta}_0\right] \tag{32b}$$

With this restructuring of the update equations, the need to estimate $\hat{\phi}$ vanishes, resulting in an effective two-component model for the rotation angle and gyro drift bias. The update of Eq. (32a) is propagated into future estimates of the angle and the drift bias, but the update of Eq. (32b) is only effective at the time of the star tracker measurement. The appearance of Eq. (32b) as the optimal combination of independent quantities $\bar{\theta}_0(+)$ and $\tilde{\theta}_0$ with standard deviations σ_e and σ_n is very misleading because these quantities are correlated by Eq. (32a).

Conclusions

Analytic expressions for the steady-state accuracy of a single-axis Kalman filter combining data from a gyro and an angle sensor have proven very useful in tailoring attitude sensor specifications to mission requirements. This Note provides a very simple modification to Farrenkopf's[5] analytic expressions to include the effects of angle white noise on the output of an RIG. The resulting equations depend on the star tracker update interval, but are independent of the gyro update interval. The three-component model developed in this Note leads to an effective two-component filter for the attitude angle and gyro drift bias that is identical with Farrenkopf's model if gyro angle output noise is absent. In the presence of gyro angle output white noise, the update of the angle estimate resulting from a star tracker measurement can be broken into two parts. The first part, which vanishes when the gyro drift parameters are zero, is propagated forward into future angle estimates. The second part of the update, which is independent of the gyro drift parameters and vanishes if the gyro readout white noise is zero, is only effective at the time of the star tracker measurement.

Acknowledgments

This Note has greatly benefited from discussions with Michael Femiano, Kong Ha, and Russell Carpenter.

References

[1] Kalman, R. E., "A New Approach to Linear Filtering and Prediction Problems," *Transactions of the ASME, Series D, Journal of Basic Engineering*, Vol. 82, No. 1, 1960, pp. 35–45.

[2] Gelb, A. (ed.), *Applied Optimal Estimation*, MIT Press, Cambridge, MA, 1974, pp. 107–112.

[3] Potter, J. E., and Vander Velde, W. E., "Optimum Mixing of Gyroscope and Star Tracker Data," *Journal of Spacecraft and Rockets*, Vol. 5, No. 5, 1968, pp. 536–540.

[4] Lefferts, E. J., Markley, F. L., and Shuster, M. D., "Kalman Filtering for Spacecraft Attitude Estimation," *Journal of Guidance, Control, and Dynamics*, Vol. 5, No. 5, 1982, pp. 417–429.

[5] Farrenkopf, R. L., "Analytic Steady-State Accuracy Solutions for Two Common Spacecraft Attitude Estimators," *Journal of Guidance and Control*, Vol. 1, No. 4, 1978, pp. 282–284.

[6] Fallon, L., III, "Gyroscope Models," *Spacecraft Attitude Determination and Control*, edited by James R. Wertz, D. Reidel, Dordrecht, The Netherlands, 1978, pp. 266–270.

Robust H_2 Estimation with Application to Robust Fault Detection

Tinglun Song* and Emmanuel G. Collins Jr.†
Florida A&M University and Florida State University, Tallahassee, Florida 32310

Nomenclature

$\mathcal{D}^n$	=	$n \times n$ real diagonal matrix
$\dim(M)$	=	dimension of M
$\boldsymbol{E}(\cdot)$	=	expectation
$\|G(z)\|_2$	=	$\left\{\frac{1}{2\pi}\int_0^{2\pi} \text{tr}[G(e^{j\theta})]^*G(e^{j\theta}))\,\text{d}\theta\right\}^{1/2}$
$M_2 > M_1$	=	$M_2 - M_1$ positive definite
$M_2 \geq M_1$	=	$M_2 - M_1$ nonnegative definite
$\mathcal{N}^n$	=	$n \times n$ nonnegative definite matrix
$\mathcal{R}, \mathcal{Z}^+$	=	real numbers, nonnegative integers
$\mathcal{R}^{m\times n}$	=	$m \times n$ real matrices
Vec$(\cdot)$	=	standard column stacking operator
$z_{i,j}$	=	(i, j) element of matrix Z

I. Introduction

FAULT detection of dynamic systems has been an active research area in recent years.[1–5] Fault detection can be achieved by using either physical redundancy or analytical redundancy, for example, estimator-based fault detection methods. The key step in estimator-based fault detection methods is to generate residuals that are accentuated by faults. These residuals are then compared with some threshold values to determine whether faults have occurred. Logically, the existence of uncertainties and disturbance inputs, that is, plant disturbances and measurement noise, obscures the effect of faults and is, therefore, a source of false alarms. To reduce false alarm rates and improve fault detection accuracy, the residuals generated should be robust against uncertainties and disturbance inputs.

Received 24 August 1999; revision received 9 May 2000; accepted for publication 20 June 2000.

*Research Assistant, Department of Mechanical Engineering; currently Project Engineer, Altair Engineering, Inc., Troy, Michigan, 48084; tinglun@altair.com.

†Professor, Department of Mechanical Engineering; ecollins@eng.fsu.edu.